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Exceptional gauge theories in 3×3 matrix formalism

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Abstract. A matrix interpretation of Tits–Vinberg construction for the Lie algebras of exceptional groups is proposed. We use 3×3 matrices over Rosenfeld algebra completed with its automorphisms which are induced via external multiplication of imaginary units. This allows one to express exceptional gauge theories in a compact and comprehensible form suitable for the purposes of model building.

1. Introduction

Recently, interesting attempts to unify strong, weak and electromagnetic interactions into a single gauge theory based upon exceptional groups have been made by Gürsey and other authors (Gürsey 1975, Gürsey *et al* 1976, Gürsey and Sikivie 1976, Sikivie and Gürsey 1977, Konstein *et al* 1977, Ramond 1976, 1977). However, exceptional groups possess high dimensions (e.g. $\dim \mathrm{E}_6 = 78$, $\dim \mathrm{E}_7 = 133$) and a huge number of structure constants respectively. Therefore in the standard gauge theory formulation where the fundamental representation of dimension N is described by an element of N-dimensional vector space, and so on, all the calculations become tedious making the results hard to comprehend. At the same time a close connection is well known between all the exceptional groups and 3×3 matrices over the Cayley algebra of octonions. Their fundamental representations can be treated compactly in terms of such matrices and this fact was used by Gürsey (1975). However, their adjoint representations also include the automorphisms of the so called Rosenfeld algebras.

In the present paper some external multiplication operation of imaginary units is introduced to represent these automorphisms. In such a way a purely matrix formulation of the Tits–Vinberg construction is achieved entirely in terms of 3×3 matrices. The formalism proposed gives a possible way of making compact exceptional gauge theories in an adequate group structure form. Both fundamental and adjoint representations are found to be generalised 3×3 matrices. The invariants take a simple form. All the calculations reduce to multiplications of these matrices and manipulations with the traces of their products. Reduction of the initial grand group to physically interesting subgroups becomes trivial.

2. An octoniai description of exceptional gauge theories

According to the famous Hurwitz theorem all the hypercomplex systems of numbers

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reduce to the algebras of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{Q}$) and octonions ($\mathbb{O}$).

Table 1.

Algebra	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}$	$\mathbb{O}$
Basis	$\mathbf{1}$	$\mathbf{1}, \mathrm{i}$	$\mathbf{1}, q_i, \quad i=1,2,3$	$\mathbf{1}, e_\alpha, \quad \alpha=1\ldots7$
Multiplication table	$\mathbf{1}.\mathbf{1}=\mathbf{1}$	$\mathrm{i}^2=-\mathbf{1}$	$q_i q_k = -\delta_{ik}+\epsilon_{ikl}q_l$	$e_\alpha e_\beta = -\delta_{\alpha\beta}+f_{\alpha\beta\gamma}e_\gamma$
Involution	—	$\mathrm{i} \overset{*}{\rightarrow} -\mathrm{i}$	$q_i \overset{*}{\rightarrow} -q_i$	$e_\alpha \overset{*}{\rightarrow} -e_\alpha$
		$\mathbf{1} \overset{*}{\rightarrow} \mathbf{1}$	$\mathbf{1} \overset{*}{\rightarrow} \mathbf{1}$	$\mathbf{1} \overset{*}{\rightarrow} \mathbf{1}$
Continuous group of automorphisms	—	—	SO(3)	G_2

In table 1, ϵ_{ikl} is a totally antisymmetric tensor of rank 3, $f_{\alpha\beta\gamma}$ are totally antisymmetric and $f_{\alpha\beta\gamma}=1$ for $\alpha\beta\gamma=(1\ 2\ 3)\ (5\ 1\ 6),\ (6\ 2\ 4),\ (4\ 3\ 5),\ (4\ 7\ 1),\ (6\ 7\ 3),\ (5\ 7\ 2)$. Let us define the quantities $e_{\alpha\beta}$ to represent infinitesimal automorphisms of $\mathbb{O}$:

$$[e_{\alpha\beta}, e_\gamma] = L_{\alpha\beta,\gamma\delta}e_\delta \tag{1}$$

$$[e_{\alpha\beta}, e_{\gamma\delta}] = C_{\alpha\beta,\gamma\delta,\mu\nu}e_{\mu\nu} \tag{2}$$

where $L_{\alpha\beta,\gamma\delta}$ is a realisation of infinitesimal automorphism on imaginary units e_α

$$L_{\alpha\beta,\gamma\delta} = 3\delta_{\alpha\gamma}\delta_{\beta\delta} - 3\delta_{\alpha\delta}\delta_{\beta\gamma} - f_{\alpha\beta\epsilon}f_{\epsilon\gamma\delta}$$

and $C_{\alpha\beta,\gamma\delta,\mu\nu}$ are structure constants of group G_2 in such a realisation.

Now we introduce an external multiplication of imaginary units

$$e_\alpha \vee e_\beta = -e_\beta \vee e_\alpha \equiv e_{\alpha\beta} \tag{3}$$

and supply the resulting set of elements $\xi = \{\mathbf{1}, e_\alpha, e_{\alpha\beta}\}$ with the structure of an algebra with involution. The latter must be in agreement with involution in $\{\mathbf{1}, e_\alpha\}$. Then transformation properties of e_α and $e_{\alpha\beta}$ under the automorphisms group together with the involution '$*$'

$$e_\alpha \overset{*}{\rightarrow} -e_\alpha, \qquad e_{\alpha\beta} \overset{*}{\rightarrow} -e_{\alpha\beta} \tag{4}$$

fix the anticommutators:

$$\{e_{\alpha\beta}, e_\gamma\} = 0 \tag{5}$$

$$\{e_{\alpha\beta}, e_{\gamma\delta}\} = L_{\alpha\beta,\gamma\delta}. \tag{6}$$

(An arbitrary constant on the right-hand side of equation (6) is chosen to be one because of the subsequent realisation of the Tits–Vinberg construction.) Further definitions

$$e_{\alpha\beta} \vee e_{\gamma\delta} = \tfrac{1}{2}[e_{\alpha\beta}, e_{\gamma\delta}] \tag{7}$$

$$e_\alpha \vee e_{\beta\gamma} = \tfrac{1}{2}[e_\alpha, e_{\beta\gamma}] \tag{8}$$

give us the construction of $\xi\mathbb{O}$ closed with respect to both usual and external multiplication. We may formally introduce an external multiplication in the remaining Hurwitz algebras and equate it to the usual commutator of imaginary units. In $\mathbb{R}$ and $\mathbb{C}$ it results identically in zero, and in $\mathbb{Q}$ it gives an element $\epsilon_{ijk}q_k$ because

$$[\epsilon_{ijk}q_k, q_l] = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})q_l$$

realises an infinitesimal automorphism of SO(3) on q_i. Now we have to consider an arbitrary Rosenfeld algebra $H_8^m \equiv A^m \otimes \mathbb{O}$,

$$H_8^1 = \mathbb{R}\otimes\mathbb{O} \equiv \mathbb{O}, \qquad H_8^2 = \mathbb{C}\otimes\mathbb{O}, \qquad H_8^4 = \mathbb{Q}\otimes\mathbb{O}, \qquad H_8^8 = \mathbb{O}\otimes\mathbb{O}.$$

Denote by $\overset{m}{q}_A$ imaginary units of A^m. Basic elements of H_8^m are

$$\mathbf{1}, e_\alpha, \overset{m}{q}_A, e_\alpha \overset{m}{q}_A, \qquad A = 1, 2 \ldots;$$

and involution is given by

$$e_\alpha \overset{*}{\to} -e_\alpha, \qquad \overset{m}{q}_A \overset{*}{\to} -\overset{m}{q}_A; \qquad e_\alpha\overset{m}{q}_A \overset{*}{\to} e_\alpha\overset{m}{q}_A.$$

The external multiplication may be introduced naturally in H_8^m by (3), (7) and (8) for $\overset{m}{q}_A$ and e_α separately and by

$$\overset{m}{q}_A \vee e_\alpha = 0 \tag{9}$$

$$(\overset{m}{q}_A e_\alpha)\vee(\overset{m}{q}_B e_\beta) \equiv \tfrac{1}{2}\{\overset{m}{q}_A\overset{m}{q}_B\}(e_\alpha \vee e_\beta) + \tfrac{1}{2}\{e_\alpha e_\beta\}(\overset{m}{q}_A \vee \overset{m}{q}_B). \tag{10}$$

Basic elements of the corresponding ξ-construction are

$$(\mathbf{1}, \overset{m}{q}_A, \overset{m}{q}_{AB}, e_\alpha, e_{\alpha\beta}, e_\alpha\overset{m}{q}_A).$$

It is clear that

$$[\overset{m}{q}_{AB} e_\alpha] = 0; \qquad [e_{\alpha\beta}, \overset{m}{q}_A] = 0. \tag{11}$$

Thus the ξ-construction over algebra H_8^m introduces its infinitesimal automorphisms in the latter. Using it one may simplify the Tits–Vinberg construction (Vinberg 1966) for the Lie algebras of exceptional groups and utilise the latter efficiently in gauge theory construction. Namely, an element of Lie algebra of exceptional group F_4, E_6, E_7, E_8 is represented by a sum of anti-Hermitian traceless matrix and infinitesimal automorphism in H_8^1, H_8^2, H_8^4 and H_8^8 respectively. ξH_8^m being introduced gives us the possibility of representing such an element as a single 3×3 anti-Hermitian matrix over ξH_8^m, which is a sum of the anti-Hermitian traceless matrix over H_8^m and some linear combination of elements $\overset{m}{q}_{AB}$ and $e_{\alpha\beta}$

$$\tilde{\mathscr{A}} = \mathscr{A} + (a_{AB}\overset{m}{q}_{AB} + a'_{\alpha\beta}e_{\alpha\beta}) \tag{12}$$

where E is the 3×3 unit matrix.

The Lie multiplication takes a simple form and is given by a generalised commutator:

$$[\tilde{\mathscr{A}}\tilde{\mathscr{B}}]g = [\tilde{\mathscr{A}}\tilde{\mathscr{B}}] - \tfrac{1}{3}\mathrm{Tr}[\tilde{\mathscr{A}}\tilde{\mathscr{B}}]\mathrm{E} + \tfrac{1}{3}\mathrm{Tr}[\tilde{\mathscr{A}} \vee \tilde{\mathscr{B}}]\mathrm{E}. \tag{13}$$

Here $[\tilde{\mathscr{A}}\tilde{\mathscr{B}}]$ is a matrix commutator, $[\tilde{\mathscr{A}} \vee \mathscr{B}]$ is a matrix commutator in which matrix elements are externally multiplied. The last term on the right-hand side of (13) is needed for validity of the Jacobi identity and represents the corresponding term in the Tits–Vinberg construction in terms of ξH_8^m.

Thus, we have a realisation of the adjoint representation of any exceptional group as a corresponding anti-Hermitian 3×3 matrix. For the purpose of gauge theory building we have to know the transformations of the fundamental representation. The following trick is convenient to obtain fundamental representations (especially having in mind 56 of E_7). Using the fact that

$$F_4 \subset E_6 \subset E_7 \subset E_8, \tag{14}$$

with the help of decomposition of anti-Hermitian in H_8^m matrix $\mathscr{A}$:

$$\mathscr{A} = A + q_A M_A \tag{15}$$

(where A is anti-Hermitian, M_A are Hermitian 3×3 matrices over octonions) it is easy to extract an adjoint representation of the lower group from the adjoint representation of any (except F_4) group in (14) and, using (13), to establish the transformational properties of the remainder under transformations of this lower group. For example, after exclusion of 52 (adjoint representation of F_4) from 78 (adjoint representation of E_6), 26 components remain, transforming according to the fundamental representation of F_4. In such a way we get the known decompositions

$$\begin{aligned} &E_6 \text{ with respect to } F_4\text{:} \quad 78 = 52 + 26 \\ &E_7 \text{ with respect to } E_6\text{:} \quad 133 = 78 + 27 + \overline{27} + 1 \\ &E_8 \text{ with respect to } E_7\text{:} \quad 248 = 133 + 56 + 56 + 1 + 1 + 1 \end{aligned} \tag{16}$$

and the following realisations of exceptional groups in fundamental representations given in table 2. Now having all that is necessary, we proceed to build the gauge E_6 theory in octonion formalism. In such a theory the fundamental fermions combine in a 27-plet which is 3×3 matrix N over H_8^2, such that $N^+ = N$. The gauge fields (vector mesons) form a 78-plet and they are represented by an anti-Hermitian matrix $\tilde{\mathscr{A}}_\mu$ over

Table 2.

Group	Adjoint representation	Dim	Fundamental representation	Dim	Realisation of group in fundamental representation
F_4	Anti-Hermitian 3×3 matrix over ξH_8^1 (see (12)) $\tilde{\mathscr{A}}_1$	52	J† $\operatorname{Tr} J = 0$	26	$\delta_{F_4} J = [\tilde{\mathscr{A}}_1, J]$
E_6	Anti-Hermitian 3×3 matrix over ξH_8^2 $\tilde{\mathscr{A}}_2$	78	$N = J_1 + \mathrm{i}J_2$	27 $\overline{27}$	$\delta_{E_6} N = \tilde{\mathscr{A}}_2 N + N\tilde{\mathscr{A}}_2^+$ $\delta_{E_6} N = \tilde{\mathscr{A}}_2^* N + N\tilde{\mathscr{A}}_2^{*+}$ ‡
E_7	Anti-Hermitian 3×3 matrix over ξH_8^4 $\tilde{\mathscr{A}}_4$ $\tilde{\mathscr{A}}_4 = \tilde{\mathscr{A}}_2 + \omega a^* + \omega^* a + q_3\phi E$ where $a = J_1 + \mathrm{i}J_2$, $\omega = \dfrac{q_1 + \mathrm{i}q_2}{2}$	133	A set (X, Y, η, ξ) where η, ξ are complex numbers, and X and Y are of form $J_1 + \mathrm{i}J_2$	56	$\delta_{E_7} X = \tilde{\mathscr{A}}_2 X + X\tilde{\mathscr{A}}_2^+ + 2a^* \times Y - ia\eta - \frac{1}{3}\mathrm{i}\phi X$ $\delta_{E_7} Y = \tilde{\mathscr{A}}_2^* Y + Y\tilde{\mathscr{A}}_2^{*+} - 2a \times X + \mathrm{i}a\xi + \frac{1}{3}\mathrm{i}\phi Y$ $\delta_{E_7}\eta = -\frac{1}{2}\mathrm{i}\operatorname{Tr}\{a^*, X\} - \mathrm{i}\phi\eta$ $\delta_{E_7}\xi = \frac{1}{2}\mathrm{i}\operatorname{Tr}\{a, Y\} + \mathrm{i}\phi\xi$§
E_8	Anti-Hermitian 3×3 matrix over ξH_8^8 $\tilde{\mathscr{A}}_8$	248	Coincides with adjoint representation $\tilde{\mathscr{A}}_8$	248	See (13)

† Here a symbol J is used for Hermitian 3×3 matrix over $\mathbb{O}$.

‡ Here and thereafter a symbol '+' denotes Hermitian conjugation in $\mathbb{O}$ while symbol '*' denotes replacement $\mathrm{i} \to -\mathrm{i}$.

§ Freudenthal product is: $a \times b \equiv \frac{1}{2}(\{a, b\} - \operatorname{Tr}(a)b - \operatorname{Tr}(b)a + (\operatorname{Tr}(a)\operatorname{Tr}(b) - \frac{1}{2}\operatorname{Tr}\{a, b\})E)$.

ξH_8^2. The covariant derivative is given by

$$D_\mu N = \partial_\mu N + e(\tilde{\mathscr{A}}_\mu N + N\tilde{\mathscr{A}}_\mu^+) \tag{17}$$

where e is equivalent to the gauge constant, spinor indices being suppressed. The covariant curl of vector field is

$$\tilde{\mathscr{F}}_{\mu\nu} = \partial_\mu \tilde{\mathscr{A}}_\nu - \partial_\nu \tilde{\mathscr{A}}_\mu + e[\tilde{\mathscr{A}}_\mu \tilde{\mathscr{A}}_\nu]g. \tag{18}$$

For representations under consideration bilinear and trilinear invariants are

$$\mathrm{Tr}\{N^*, M\} \tag{19a}$$

$$\mathrm{Tr}\{N^*, (\tilde{\mathscr{A}}M + M\tilde{\mathscr{A}}^+)\} \tag{19b}$$

$$\mathrm{Tr}(\tilde{\mathscr{A}}\tilde{\mathscr{B}} + \tilde{\mathscr{B}}^*\tilde{\mathscr{A}}^*). \tag{19c}$$

Here N, M are 27-plets, $\tilde{\mathscr{A}}, \tilde{\mathscr{B}}$ are 78-plets. Using equation (19) the invariant Lagrangian is written in the compact form

$$\mathscr{L} = \mathrm{Tr}\{N^*, \not{D}N\} + \mathrm{Tr}(\tilde{\mathscr{F}}_{\mu\nu}\tilde{\mathscr{F}}^{\mu\nu} + \tilde{\mathscr{F}}^{\mu\nu *}\tilde{\mathscr{F}}^*_{\mu\nu}) \tag{20}$$

$\not{D} \equiv \gamma_\mu D_\mu$, spinor indices being suppressed. Analogously it is easy to introduce the scalar field, transforming as 27 or 78 in the theory. The trilinear invariant

$$\mathrm{Tr}\{\mathscr{P} \times \mathscr{P}), \mathscr{P}\}, \mathscr{P} = 27 \tag{21}$$

is useful for constructing its Higgs self-interaction together with (19a) and its square. This very compact and elegant form of the E_6 gauge theory can be translated into the customary language of complex numbers using the so called split basis of the Cayley algebra:

$$u_\alpha = \frac{e_\alpha + \mathrm{i}e_{\alpha+3}}{2}, \qquad u_\alpha^* = \frac{e_\alpha - \mathrm{i}e_{\alpha+3}}{2}$$
$$u_0 = \frac{1+\mathrm{i}e_7}{2}; \qquad u_0^* = \frac{1-\mathrm{i}e_7}{2} \qquad \alpha = 1, 2, 3. \tag{22}$$

The advantages of such a basis were mentioned by Gürsey and Günaydin (1973) (see the corresponding multiplication table therein). We should remark only that u_α and u_α^* transform as a triplet and anti-triplet respectively under $\mathrm{SU^c(3)}$. The latter is the subgroup of the automorphism group of octonions which leaves e_7 invariant and which is identified with the colour symmetry in the theories under consideration.

Introducing the split-basis in fundamental and adjoint representations one comes to the following decompositions of matrices over octonions

$$N = Mu_0^* + M^{\mathrm{T}}u_0 + M_\alpha u_\alpha^* + N_\alpha u_\alpha \tag{23}$$

where M is a usual (complex) matrix of general form, 'T' denotes transposition of the matrix, N_α and M_α are complex matrices of the form

$$\begin{pmatrix} 0 & \omega_\alpha & \mu_\alpha \\ -\omega_\alpha & 0 & \nu_\alpha \\ -\mu_\alpha & -\nu_\alpha & 0 \end{pmatrix}$$

$$\tilde{\mathscr{A}} = Bu_0^* + \tilde{B}u_0 + A_\alpha u_\alpha^* + A_\alpha^+ u_\alpha + \underset{i,k\neq 7}{G_{ik}e_{ik}} + \theta_k e_{7k}E; \qquad i, k = 1, \ldots, 7 \tag{24}$$

where B and $\tilde{B}$ are anti-Hermitian traceless complex matrices, A_α is a traceless complex matrix of general form, and θ_{ik}, θ_k are real; '+' with respect to usual matrices means Hermitian conjugation. Known decompositions of E_6 with respect to the maximal subgroup $SU(3)\otimes SU(3)\otimes SU^c(3)$

$$\begin{aligned}27&=(3.\bar{3}.1^c)+(\bar{3}.1.\bar{3}^c)+(1.3.3^c)\\78&=(8.1.1^c)+(1.8.1^c)+(3.3.\bar{3}^c)+(\bar{3}.\bar{3}.3^c)+(1.1.8^c)\end{aligned}\tag{25}$$

are derived immediately from definitions (13) and table 2, provided the generators of $SU(3)_{1(2)}$ groups (remaining after extraction of $SU^c(3)$) are

$$\begin{aligned}&SU(3)_1\colon \frac{e_7+i}{2}\lambda_a = i\,u_0^*\lambda_a \qquad \lambda_a - \text{Gell-Mann matrices}\\&SU(3)_2\colon \frac{e_7-i}{2}\lambda_a = -i u_0\lambda_a \qquad a=1,2,\ldots,8.\end{aligned}\tag{26}$$

According to these decompositions M transforms as $(3.\bar{3}.1^c)$, M_α as $(\bar{3}.1.\bar{3}^c)$, N_α as $(1.3.3^c)$, B as $(8.1.1^c)$, $\tilde{B}$ as $(1.8.1^c)$, G_{ik} as $(1.1.8^c)$ and $\tilde{A}_\alpha$ as $(3.3.\bar{3}^c)$ and $\tilde{A}_\alpha^+$ as $(\bar{3}.\bar{3}.3^c)$, $\tilde{A}_\alpha \equiv A_\alpha + \frac{1}{3}i\theta_\alpha E$ respectively. Here

$$\theta_k e_{7k} \equiv \tilde{\theta}_\alpha \nu_\alpha^* + \tilde{\theta}_\alpha^* \nu_\alpha, \qquad \tilde{\theta}_\alpha \equiv \theta_\alpha + i\theta_{\alpha+3}, \quad \nu_\alpha \equiv e_{7\alpha} + ie_{7\alpha+3}, \quad \alpha = 1,2,3.$$

The interaction Lagrangian in terms of these decompositions is written as

$$\begin{aligned}\mathscr{L}_{\text{int}} = e\,\mathrm{Tr}(&M + BM + M^*\tilde{B}M^{\mathrm{T}} + M_\alpha^+\tilde{A}_\alpha M^{\mathrm{T}} + N_\alpha^+ A_\alpha^+ M^{\mathrm{T}}\\&- M_\alpha^* B M_\alpha - N_\alpha^*\tilde{B}N_\alpha - \tfrac{1}{2}M_\alpha^* \mathcal{G}_A \Lambda_{\alpha\beta}^A M_\beta\\&- \tfrac{1}{2}N_\alpha^* \mathcal{G}_A\Lambda_{\alpha\beta}^A N_\beta - M_\alpha^*\tilde{A}_\beta^+ N_\gamma\epsilon_{\alpha\beta\gamma} + N_\alpha^*\tilde{A}_\beta M_\gamma \epsilon_{\alpha\beta\gamma}) + \text{HC}\end{aligned}\tag{27}$$

$(A=1,2,\ldots 8)$ where we defined $G_{ik}e_{ik}u_\alpha \equiv G_A\Lambda_{\alpha\beta}^A u_\beta$, $\Lambda_{\alpha\beta}^A$ is a realisation of $SU^c(3)$ over U_α; '*' over the matrix means complex conjugation, $B \equiv \gamma_\mu B_\mu$ etc. If electric charge in such a theory is defined as

$$Q \equiv Q_1 + Q_2 \tag{28}$$

where $Q_{1(2)}$ is the electric charge operator in $SU(3)_{1(2)}$ (cf Gürsey 1975), then matrix M in the 27-plet of fermions will represent leptons, N_α and M_α are quarks and anti-quarks. Among vector mesons we find intermediate bosons of the weak interaction B_μ and $\tilde{B}_\mu$, gluons $G_{A\mu}$ which mediate strong interaction and lepto-quarks $A_{\alpha\mu}$ whose interactions do not conserve baryon number. The structure of all interactions is evident from (27). Now we shall discuss briefly the E_7 theory, which is most interesting in view of the applications. Note that the known decompositions with respect to maximal subgroup $SU(6)\otimes SU^c(3)$ can be easily obtained directly from (13) and table 2.

$$\begin{aligned}56&=(20.1^c)+(6.3^c)+(\bar{6}.\bar{3}^c)\\133&=(35.1^c)+(15.\bar{3}^c)+(\overline{15}.3^c)+(1.8^c).\end{aligned}\tag{29}$$

In comparison with the above E_6 scheme E_7 theory contains a richer spectrum of leptons, additional quarks, lepto-quarks and intermediate bosons. The kinetic term of the theory is built with the help of invariant

$$\xi^*\xi + \eta^*\eta + \mathrm{Tr}\,\tfrac{1}{2}\{X^*X\} + \mathrm{Tr}\,\tfrac{1}{2}\{Y^*Y\}.\tag{30}$$

The unitary transformation, connecting representation 56* and 56, can be easily found

$$56^* \to (-Y^*, X^*, -\xi^*, \eta^*) = 56. \tag{31}$$

(This reflects the pseudoreality of this representation.) Hence, another form of this invariant exists:

$$\xi_1\eta_2 - \xi_2\eta_1 + \mathrm{Tr}\,\tfrac{1}{2}\{X_1, Y_2\} - \mathrm{Tr}\,\tfrac{1}{2}\{X_2, Y_1\} \tag{32}$$

$(X_i, Y_i, \xi_i, \eta_i)$, $i = 1,2$ are two 56-plets. An interaction is given by invariant

$$\xi^*\Delta_{\mathrm{E}_7}\xi + \eta^*\Delta_{\mathrm{E}_7}\eta + \mathrm{Tr}\,\tfrac{1}{2}\{X^*, \Delta_{\mathrm{E}_7}X\} + \mathrm{Tr}\,\tfrac{1}{2}\{Y^*, \Delta_{\mathrm{E}_7}Y\} + \mathrm{HC} \tag{33}$$

where Δ_{E_7} is the E_7 transformation given in table 2 in which all parameters are replaced by corresponding gauge fields which are multiplied by γ matrices. The kinetic term and interaction for vector fields are built with the help of invariant

$$\mathrm{Tr}(\tilde{\mathscr{A}}_4\mathscr{B}_4 + \tilde{\mathscr{B}}_4^*\mathscr{A}_4^*) \tag{34}$$

(# denotes quaternion conjugation), in complete analogy with E_6 theory. Again, the introduction of scalar 56-plet or 133-plet is possible. Its self-interaction is given in the first case by the invariant

$$\mathrm{Tr}\,\tfrac{1}{2}\{X\times X, Y\times Y\} - \xi\mathrm{Tr}\,\tfrac{1}{2}\{X\times X, X\} - \eta\mathrm{Tr}\,\tfrac{1}{2}\{Y\times Y, Y\} - \tfrac{1}{4}(\mathrm{Tr}\,\tfrac{1}{2}\{XY\} - \xi\eta)^2 \tag{35}$$

(cf Jacobson 1971). To understand the structure of the theory we need the interaction Lagrangian in a reduced form. Making use of decompositions (notation is the same as in table 2)

$$\begin{aligned} X &= Mu_0^* + M^{\mathrm{T}}u_0 + M_\alpha u_\alpha^* + N_\alpha u_\alpha \\ Y &= Pu_0^* + P^{\mathrm{T}}u_0 + P_\alpha u_\alpha^* + T_\alpha u_\alpha \\ a &= a_0u_0^* + a_0^{\mathrm{T}}u_0 + a_\alpha u_\alpha^* + \tilde{a}_\alpha u_\alpha \end{aligned} \tag{36}$$

we obtain the Lagrangian of interactions as a sum of terms (27) for X and Y separately and the following expression carrying new peculiar E_7 interactions

$$\begin{aligned} \mathscr{L}_{\mathrm{int}\,\mathrm{E}_7} = e\mathrm{Tr}[&M^+a_0^+P + M^*a_0^*P^{\mathrm{T}} - M^*a_\alpha^*P_\alpha - M^+\tilde{a}_\alpha^*T_\alpha - M_\alpha^*a_0^+P_\alpha - M_\alpha^*\tilde{a}_\alpha^*P^{\mathrm{T}} \\ &+ \epsilon_{\alpha\beta\gamma}M_\alpha^*a_\beta^*T_\gamma - N_\alpha^*a_0^*T_\alpha - N_\alpha^*a_\alpha^*P + \epsilon_{\alpha\beta\gamma}N_\alpha^*\tilde{a}_\beta^*P_\gamma - \mathrm{Tr}(M)a_0^+P \\ &+ \tfrac{1}{2}\mathrm{Tr}(M)a_\alpha^*P_\alpha + \tfrac{1}{2}\mathrm{Tr}(M)\tilde{a}_\alpha^*T_\alpha + \mathrm{Tr}(M^*)\mathrm{Tr}(a_0^*)P - M^+P\mathrm{Tr}(a_0^+) \\ &+ \tfrac{1}{2}M_\alpha^*P_\alpha\mathrm{Tr}(a_0^+) + \tfrac{1}{2}N_\alpha^*T_\alpha\mathrm{Tr}(a_0^+) - M^+a_0^+\mathrm{Tr}(P) - \tfrac{1}{2}M_\alpha^*\tilde{a}_\alpha^*\mathrm{Tr}(P) \\ &- \tfrac{1}{2}N_\alpha^*a_\alpha\mathrm{Tr}(P) - \tfrac{1}{3}\mathrm{i}\phi M^+M + \tfrac{1}{6}\mathrm{i}\phi(M_\alpha^*M_\alpha + N_\alpha^*N_\alpha) - \mathrm{i}\eta(\{M^+a_0\} \\ &- (M_\alpha^*a_\alpha + N_\alpha^*\tilde{a}_\alpha)] + (X\to Y, a^*\to -a, \eta\to -\xi) + \mathrm{i}\bar{\xi}\phi\xi + \tfrac{1}{2}\mathrm{i}\bar{\xi}\mathrm{Tr}[\{a_0P\} \\ &- (\tilde{a}_\alpha P_\alpha - a_\alpha T_\alpha)] - \mathrm{i}\bar{\eta}\phi\eta - \tfrac{1}{2}\mathrm{i}\bar{\eta}\mathrm{Tr}[\{a_0^+M\} - (a_\alpha^*M_\alpha + \tilde{a}_\alpha^*N_\alpha)] + \mathrm{HC}. \end{aligned} \tag{37}$$

Here $(X\to Y, a^*\to -a, \eta\to -\xi)$ denotes terms obtained from the previous ones by these replacements.

The Lagrangian (37) has a clear structure and fixes all the new couplings in the theory in comparison with that of E_7. Its length reflects a richer symmetry in the E_7 case.

3. Conclusion

Thus, it has been shown that the 3×3 matrix formalism gives the exceptional gauge theories in compact and clear form and essentially simplifies their treatment. The evident structure of the theory in such a formalism makes the task of physical model building much more easy and clarifies the problem of assignment. These subjects will be discussed elsewhere.

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